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Maximal Buchsbaum modules over Gorenstein local rings

Mutsumi Amasaki

Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima 739-8524, Japan

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Abstract

We clarify the structure of a maximal Buchsbaum module over a Gorenstein local ring by resolving the dual of its minimal free resolution into the mapping cones of successive chain maps from free complexes to direct sums of finite copies of the minimal free resolution of the residue field.

Keywords: Buchsbaum module; Free resolution

Introduction

A noetherian module M over a local ring $(A, \mathfrak{m})$ of dimension d is called maximal if $\dim(M) = d$. When M is a maximal Buchsbaum module and A is regular, we know by Goto's structure theorem that M is the direct sum of finite copies of the syzygy modules of $k := A/\mathfrak{m}$ over A and such direct sum decomposition is unique (see [5]). Although this result is restricted to the maximal case, it is very useful in studying ideals defining Buchsbaum rings. In fact, applying it to Bourbaki sequences, we can construct and classify graded Buchsbaum integral domains of codimension two (see [1, Section 7]). Its combination with analysis of Gröbner basis with respect to generic coordinates enables us to

E-mail address: amasaki@hiroshima-u.ac.jp.

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classify roughly all homogeneous ideals in polynomial rings defining graded Buchsbaum rings (see [1, Sections 5 and 6]).

Over regular local rings, to say that M is Buchsbaum is the same thing as saying that M is surjective-Buchsbaum, but generally these two notions do not coincide (see [6, Definition 2.1], [7] and [8]). So far, an enhancement of Goto's theorem has been given only for surjective-Buchsbaum case by the work of Kawasaki [6]. More precisely, if M is a maximal surjective-Buchsbaum module of finite injective dimension over a Cohen–Macaulay local ring, then it is the direct sum of finite copies of $d+1$ kinds of maximal surjective-Buchsbaum modules which are obtained by taking the tensor products of the canonical module of A with the cokernels of the dual of the minimal free resolution of k over A, and such direct sum decomposition is unique (see [6, Theorem 3.1]).

One of the important properties of Buchsbaum modules is that $\mathfrak{m}H^i_{\mathfrak{m}}(M)=0$ for all $i<\dim(M)$. If M satisfies this condition, it is called quasi-Buchsbaum. A quasi-Buchsbaum module is not necessarily Buchsbaum (see [4] for instance), so that we cannot expect that a maximal module M is always the direct sum of finite copies of some simpler ones in the quasi-Buchsbaum case, even if A is regular. When A is Gorenstein, however, we can grasp all maximal quasi-Buchsbaum modules by considering the structure of their minimal free resolutions. Assume that M is maximal with no free direct summand and let

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a minimal free resolution of M over A and

$$\cdots \xrightarrow{\delta^\vee_{-2}} P^\vee_{-2} \xrightarrow{\delta^\vee_{-1}} P^\vee_{-1} \xrightarrow{\delta^\vee_0} P^\vee_0 \xrightarrow{\delta^\vee_1} \operatorname{Im}(\delta^\vee_1) \longrightarrow 0$$

a minimal free resolution of $\operatorname{Im}(\delta^\vee_1)$ over A. We put $G_i=P^\vee_{d-i}$, $\gamma_i=\delta^\vee_{d-i+1}$ for $i\in\mathbf{Z}$ and let $(G_\bullet,\gamma_\bullet)$ be the minimal complex thus obtained (cf. [2, (4.2)]). Then M is quasi-Buchsbaum if and only if $G_\bullet$ is the minimal part of a complex $E_\bullet$ constructed in the following manner (see [2, (1.1)]). Define chain maps

$$\lambda_{0,\bullet}: F_\bullet \to (L_{\bullet-1})^{p_0},$$
$$\lambda_{j,\bullet}: \operatorname{con}(\lambda_{j-1,\bullet})_\bullet \to (L_{\bullet-j-1})^{p_j} \quad (1\leqslant j\leqslant d-1)$$

inductively and let $E_\bullet := \operatorname{con}(\lambda_{d-1,\bullet})_\bullet$, where

$$\cdots \to L_d \to \cdots \to L_1 \to L_0 \to k \to 0$$

is a minimal free resolution of the residue field k over A, $L_i=0$ for $i<0$, $F_\bullet$ is a minimal free exact complex, and $\operatorname{con}(\lambda_{j-1,\bullet})_\bullet$ denotes the mapping cone of $\lambda_{j-1,\bullet}$ (see [2, (1.6)]).

When is M Buchsbaum? The main purpose of this paper is to give an answer to this question in terms of the chain maps $\lambda_{j,\bullet}$ $(0\leqslant j\leqslant d-1)$. To state our results precisely, let $K_\bullet$ denote the Koszul complex of $x_1,\dots,x_r$ with respect to A, where $x_1,\dots,x_r$ are minimal generators of $\mathfrak{m}$. Observe that $K_\bullet$ is a subcomplex of $L_\bullet$ such that $L_\bullet/K_\bullet$ is free (see Lemma 1.1). With this notation, M is Buchsbaum if and only if we can choose

$\lambda_{j,\bullet}$ $(0 \leqslant j \leqslant d-1)$ so that $U_{j-1,\bullet} := \bigoplus_{l=0}^{j-1}(K_{\bullet-l})^{p_l}$ is a subcomplex of $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ with $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet/U_{j-1,\bullet}$ free for each $1 \leqslant j \leqslant d$ and $\lambda_{j,\bullet}|_{U_{j-1,\bullet}} = 0$ for all $1 \leqslant j \leqslant d-1$. See Definition 1.3, Theorems 1.14 and 2.4.

1. Buchsbaum cones

Throughout this paper we denote by $(A, \mathfrak{m})$ a local Gorenstein ring of dimension $d > 0$ with $k := A/\mathfrak{m}$ and $r := \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Let

$$\cdots \to L_d \to \cdots \to L_1 \to L_0 \to k \to 0$$

be a minimal free resolution of the residue field k over A. We extend this resolution by setting $L_i = 0$ for $i < 0$ and denote the resulting complex by $L_\bullet$. For minimal generators $x_1, \ldots, x_r$ of $\mathfrak{m}$, let $K(x_1, \ldots, x_l)_\bullet$ denote the Koszul complex of $x_1, \ldots, x_l$ with respect to A for $0 \leqslant l \leqslant r$ and $K_\bullet := K(x_1, \ldots, x_r)_\bullet$.

Lemma 1.1. *With the notation above, we may think of $K_\bullet$ as a subcomplex of $L_\bullet$ such that $L_\bullet/K_\bullet$ is free over A.*

Proof. Let $\partial_\bullet^L$ (respectively $\partial_\bullet^K$) denote the differential of $L_\bullet$ (respectively $K_\bullet$). Since $H_0(K_\bullet) \cong k$ and $x_1, \ldots, x_r$ are minimal generators of $\mathfrak{m}$, there is a chain map $\mu_\bullet : K_\bullet \to L_\bullet$ such that μ_0 and μ_1 are bijective. Moreover, since the homomorphism $\bar{\partial}_i^K : K_i \otimes k \to \mathfrak{m}K_{i-1}/\mathfrak{m}^2K_{i-1} = K_{i-1} \otimes (\mathfrak{m}/\mathfrak{m}^2)$ induced from ∂_i^K is injective, it follows inductively from the commutative diagram

$$\begin{tikzcd}
L_i \otimes k \arrow[r, "\bar{\partial}_i^L"] & \mathfrak{m}L_{i-1}/\mathfrak{m}^2L_{i-1} \arrow[r, equal] & L_{i-1} \otimes \big(\mathfrak{m}/\mathfrak{m}^2\big) \\
K_i \otimes k \arrow[r, "\bar{\partial}_i^K"] \arrow[u, "\mu_i \otimes k"] & \mathfrak{m}K_{i-1}/\mathfrak{m}^2K_{i-1} \arrow[r, equal] \arrow[u] & K_{i-1} \otimes \big(\mathfrak{m}/\mathfrak{m}^2\big) \arrow[u, "\mu_{i-1} \otimes (\mathfrak{m}/\mathfrak{m}^2)"']
\end{tikzcd}$$

that $\mu_i \otimes k$ is injective for all $i \in \mathbf{Z}$. In consequence, $L_\bullet/\mu_\bullet(K_\bullet)$ is a complex of free A-modules. $\square$

We fix an injective chain map $\mu_\bullet : K_\bullet \to L_\bullet$ such that $L_\bullet/\mu_\bullet(K_\bullet)$ is a complex of free A-modules, and regard $K_\bullet$ as a subcomplex of $L_\bullet$ by this embedding.

Definition 1.2. Let $F_\bullet$ be a minimal exact complex of finite free A-modules, namely a complex of finitely generated free A-modules satisfying $\mathrm{Im}(\partial_i) \subset \mathfrak{m}F_{i-1}$ and $H_i(F_\bullet) = 0$ for all $i \in \mathbf{Z}$, where $\partial_\bullet$ is the differential of $F_\bullet$. Let further m be an integer with $1 \leqslant m \leqslant d$ and p_i $(0 \leqslant i \leqslant m-1)$ nonnegative integers. We say that a complex $E_\bullet$ of A-modules is a *quasi-Buchsbaum cone* with base $(F_\bullet, p_0, \ldots, p_{m-1})$, if there are inductively defined chain maps

$$\lambda_{0,\bullet} : F_\bullet \to (L_{\bullet-1})^{p_0},$$
$$\lambda_{j,\bullet} : \mathrm{con}(\lambda_{j-1,\bullet})_\bullet \to (L_{\bullet-j-1})^{p_j} \quad (1 \leqslant j \leqslant m-1)$$

and $E_\bullet = \mathrm{con}(\lambda_{m-1,\bullet})_\bullet$, where $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ denotes the mapping cone of $\lambda_{j-1,\bullet}$.

Definition 1.3. Let $F_\bullet$, m and p_i $(0 \leqslant i \leqslant m-1)$ be the same as in Definition 1.2. Let further $E_\bullet$ be a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{m-1})$ and $\lambda_{j,\bullet}$ $(0 \leqslant j \leqslant m-1)$ be the chain mappings which $E_\bullet$ is made of. Put $U_{j,i} := \bigoplus_{l=0}^{j} (K_{i-l})^{p_l} \subset F_i \oplus (\bigoplus_{l=0}^{j} (L_{i-l})^{p_l}) = \mathrm{con}(\lambda_{j,\bullet})_i$ for $0 \leqslant j \leqslant m-1$. We say that $E_\bullet$ is a *Buchsbaum cone* with base $(F_\bullet, p_0, \dots, p_{m-1})$, if $\lambda_{j,i}|_{U_{j-1,i}} = 0$ for all $1 \leqslant j \leqslant m-1$, $i \in \mathbf{Z}$.

When $E_\bullet$ is a Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{m-1})$, the complex $U_{j-1,\bullet} := \bigoplus_{l=0}^{j-1} (K_{\bullet-l})^{p_l}$ is a subcomplex of $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet$ such that $\mathrm{con}(\lambda_{j-1,\bullet})_\bullet / U_{j-1,\bullet}$ is free for each $1 \leqslant j \leqslant m$ and $\lambda_{j,\bullet}|_{U_{j-1,\bullet}} = 0$ for all $1 \leqslant j \leqslant m-1$, with the notation of the above definitions. We have $H_i(E_\bullet) \cong k^{p_i}$ and $\mathfrak{m} H_i(E_\bullet) = 0$ for all $0 \leqslant i < m$ for a quasi-Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ by repeated use of [2, (1.4)]. On the other hand, $\mathfrak{m} H_i(E_\bullet \otimes_A A/(z_{i_1}, \dots, z_{i_n})) = 0$ for all $0 \leqslant i < m$, $0 \leqslant n \leqslant d$ and $1 \leqslant i_1 < \dots < i_n \leqslant r$, if $E_\bullet$ is a Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{m-1})$, where $z_1, \dots, z_r$ is an A-basis of $\mathfrak{m}$ (see Corollary 1.12). The purpose of the present section is to discuss this property in detail as a characterization of Buchsbaum cones. Our results obtained here will be used in the next section to describe the structure of maximal Buchsbaum modules.

Let $z_1, \dots, z_r$ be an A-basis of $\mathfrak{m}$, namely a system of minimal generators of $\mathfrak{m}$ such that $z_{i_1}, \dots, z_{i_d}$ form a system of parameters of A for every sequence $i_1, \dots, i_d$ of integers with $1 \leqslant i_1 < \dots < i_d \leqslant r$ (see [9, Chapter I, Definition 1.7]). Recall that such $z_1, \dots, z_r$ always exist by Proposition 1.9 loc. cit. In the following argument, we set $\mathfrak{B} := \{(z_{\sigma(1)}, \dots, z_{\sigma(r)}) \mid \sigma \in \mathfrak{S}_r\} \subset \mathfrak{m}^{\oplus r}$ and assume that $x_1, \dots, x_r$ are minimal generators of $\mathfrak{m}$ obtained by permuting $z_1, \dots, z_r$, i.e., $(x_1, \dots, x_r) \in \mathfrak{B}$, where $\mathfrak{S}_r$ denotes the symmetric group on r letters.

Given an A-module E and an integer n with $0 \leqslant n \leqslant d$, we will denote by ${}_{(n)}E$ the module $E/(x_{r-n+1}, \dots, x_r)E$. Further, for a complex $(S_\bullet, \varphi_\bullet)$ of A-modules, we will denote by $({}_{(n)}S_\bullet, {}_{(n)}\varphi_\bullet)$ the complex obtained by tensoring S_i and φ_i with ${}_{(n)}A$ over A for all $i \in \mathbf{Z}$. By the property of the differential of a Koszul complex, we have

$${}_{(n)}K_\bullet = \bigoplus_{j=0}^{n} \bigoplus_{r-n+1 \leqslant s_1 < \dots < s_j \leqslant r} {}_{(n)}K(x_1, \dots, x_{r-n})_{\bullet-j} \wedge \chi_{s_1} \wedge \dots \wedge \chi_{s_j},$$

where $\{\chi_1, \dots, \chi_r\}$ is the free basis of K_1 corresponding to $x_1, \dots, x_r$. Let

$$K_\bullet''^{i,n} := \bigoplus_{r-n+1 \leqslant s_1 < \dots < s_{i-1} \leqslant r} {}_{(n)}K(x_1, \dots, x_{r-n})_{\bullet-(i-1)} \wedge \chi_{s_1} \wedge \dots \wedge \chi_{s_{i-1}},$$
$$K_\bullet'''^{i,n} := \bigoplus_{r-n+1 \leqslant s_1 < \dots < s_i \leqslant r} {}_{(n)}K(x_1, \dots, x_{r-n})_{\bullet-i} \wedge \chi_{s_1} \wedge \dots \wedge \chi_{s_i}$$

for $0 \leqslant n \leqslant d,\ i \in \mathbf{Z},\ l \geqslant 0$, where we understand $K''^{i,n}_{\bullet} = 0$ (respectively $K'''^{i,n}_{\bullet} = 0$) if $n < i - 1$ or $i < 1$ (respectively $n < i$ or $i < 0$). These complexes are subcomplexes of ${}_{(n)}K_{\bullet}$. Further we denote the direct sum of the remaining summands of ${}_{(n)}K_{\bullet}$ by $K'^{\,i,n}_{\bullet}$, more precisely,

$$K'^{\,i,n}_{\bullet} := \bigoplus_{0 \leqslant j < i-1,\, i < j \leqslant n} \ \bigoplus_{r-n+1 \leqslant s_1 < \cdots < s_j \leqslant r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet - j} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_j}.$$

We have

$$K''^{i,n}_{\bullet} = K'''^{i-1,n}_{\bullet}, \qquad {}_{(n)}K_{\bullet} = K'^{\,i,n}_{\bullet} \oplus K''^{i,n}_{\bullet} \oplus K'''^{i,n}_{\bullet}.$$

Moreover $K''^{i,n}_{j} = 0$ (respectively $K'''^{i,n}_{j} = 0$) if $j < i - 1$ (respectively $j < i$). Since $K_{\bullet}$ is a subcomplex of $L_{\bullet}$ by Lemma 1.1, there are natural inclusions of $K''^{i,n}_{\bullet}$ and $K'''^{i,n}_{\bullet}$ into ${}_{(n)}L_{\bullet}$.

Lemma 1.4. *For all $0 \leqslant n \leqslant d,\ i \in \mathbf{Z}$, the inclusion map $K'''^{i,n}_{\bullet} \hookrightarrow {}_{(n)}L_{\bullet}$ (respectively $K''^{i,n}_{\bullet} \hookrightarrow {}_{(n)}L_{\bullet}$) yields a bijection $H_i(K'''^{i,n}_{\bullet}) \to H_i({}_{(n)}L_{\bullet})$ (respectively $H_{i-1}(K''^{i,n}_{\bullet}) \to H_{i-1}({}_{(n)}L_{\bullet})$).*

Proof. Let $\rho'''_{\bullet} : K'''^{i,n}_{\bullet} \to {}_{(n)}L_{\bullet}$ denote the inclusion and $H(\rho'''_i) : H_i(K'''^{i,n}_{\bullet}) \to H_i({}_{(n)}L_{\bullet})$ the natural homomorphism it induces. Since $K'''^{i,n}_{\bullet}$ is a subcomplex of ${}_{(n)}L_{\bullet}$ with $\operatorname{Ker}({}_{(n)}\partial^L_i|_{K'''^{i,n}_i}) \cong K'''^{i,n}_i$, $\operatorname{Im}({}_{(n)}\partial^L_{i+1}|_{K'''^{\ i,n}_{i+1}}) \cong \mathfrak{m}K'''^{i,n}_i$ and since ${}_{(n)}L_{\bullet}$ is minimal, we find that $H_i(K'''^{i,n}_{\bullet}) \cong k^{\binom{n}{i}}$ and that $H(\rho'''_i)$ is injective, where $\binom{n}{i} = 0$ for $i < 0,\ i > n$. On the other hand, since $L_{\bullet}$ is a minimal free resolution of $k = A/\mathfrak{m}$ over A and $x_{r-n+1}, \ldots, x_r$ is A-regular, we have $H_i({}_{(n)}L_{\bullet}) = \operatorname{Tor}^A_i(k, {}_{(n)}A) \cong k^{\binom{n}{i}}$. Hence $H(\rho'''_i)$ is bijective. Since $K''^{i,n}_{\bullet} = K'''^{i-1,n}_{\bullet}$, the same holds for the homomorphism $H_{i-1}(K''^{i,n}_{\bullet}) \to H_{i-1}({}_{(n)}L_{\bullet})$. $\square$

Lemma 1.5. *For all $0 \leqslant n \leqslant d,\ i \in \mathbf{Z}$, we have $K'''^{\ i,n}_{i-1} = 0$, $H_i({}_{(n)}L_{\bullet}/K'''^{i,n}_{\bullet}) = 0$ and $H_i({}_{(n)}L_{\bullet}/(K''^{i,n}_{\bullet} \oplus K'''^{i,n}_{\bullet})) = 0$.*

Proof. Note first $H_j(K'''^{i,n}_{\bullet}) = 0$ for $j < i$ and $H_{i-1}({}_{(n)}L_{\bullet}) \cong H_{i-1}({}_{(n)}L_{\bullet}/K'''^{i,n}_{\bullet})$ since $K'''^{i,n}_{j} = 0$ for $j < i$ by definition. With the notation of the proof of the preceding lemma, the homomorphism $H(\rho'''_i)$ is bijective in the long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_i\big(K'''^{i,n}_{\bullet}\big) &\xrightarrow{H(\rho'''_i)} H_i({}_{(n)}L_{\bullet}) \longrightarrow H_i\big({}_{(n)}L_{\bullet}/K'''^{i,n}_{\bullet}\big) \\ &\longrightarrow H_{i-1}\big(K'''^{i,n}_{\bullet}\big) = 0 \longrightarrow \cdots, \end{aligned}$$

so that $H_i({}_{(n)}L_{\bullet}/K'''^{i,n}_{\bullet}) = 0$. Let $\rho''_{\bullet} : K''^{i,n}_{\bullet} \to {}_{(n)}L_{\bullet}$ be the inclusion. By the bijectivity of $H(\rho''_{i-1})$ in the long exact sequence

$$\cdots \longrightarrow H_i\big(K_\bullet^{\prime\prime i,n}\big) \longrightarrow 0 = H_i\big({}_{(n)}L_\bullet/K_\bullet^{\prime\prime\prime i,n}\big) \longrightarrow H_i\big({}_{(n)}L_\bullet/\big(K_\bullet^{\prime\prime i,n} \oplus K_\bullet^{\prime\prime\prime i,n}\big)\big)$$
$$\longrightarrow H_{i-1}\big(K_\bullet^{\prime\prime i,n}\big) \xrightarrow{H(\rho''_{i-1})} H_{i-1}({}_{(n)}L_\bullet) \cong H_{i-1}\big({}_{(n)}L_\bullet/K_\bullet^{\prime\prime\prime i,n}\big) \longrightarrow \cdots,$$

we find $H_i({}_{(n)}L_\bullet/(K_\bullet^{\prime\prime i,n} \oplus K_\bullet^{\prime\prime\prime i,n})) = 0$. $\square$

In addition to the above lemmas, we need some more preparations to perform inductive argument for our characterization of Buchsbaum cones. All complexes treated below are complexes of finitely generated A-modules unless otherwise specified.

Lemma 1.6. *Let $(S_\bullet, \varphi_\bullet)$ be a free complex, $(T_\bullet, \psi_\bullet)$ a minimal free complex, $\lambda_\bullet : S_\bullet \to T_{\bullet-1}$ a chain map, and $C_\bullet := \operatorname{con}(\lambda_\bullet)_\bullet$ its mapping cone. Let further $U_\bullet$, $U'_\bullet$, $U''_\bullet$, $U'''_\bullet$ be free subcomplexes of $S_\bullet$ such that $U_\bullet = U'_\bullet \oplus U''_\bullet \oplus U'''_\bullet$ and $S_\bullet/U_\bullet$ is free. For an integer m, suppose that $T_{i-2} = 0$, $U''_{i-2} = 0$, $U'''_{i-1} = 0$ for $i \leqslant m$, $H_m(S_\bullet/(U''_\bullet \oplus U'''_\bullet)) = 0$, $\operatorname{Im}(\varphi_m|_{U''_m}) = \mathfrak{m}U''_{m-1}$, and that $\varphi_m(e_1), \ldots, \varphi_m(e_s)$ $(\mathrm{mod}\ \mathfrak{m}^2)$ are linearly independent elements of $U''_{m-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ over $A/\mathfrak{m}$, where $e_1, \ldots, e_s$ are free bases of U''_m. If $\mathfrak{m}H_{m-1}(C_\bullet) = 0$ and $\lambda_m|_{U'''_m} \equiv 0$ $(\mathrm{mod}\ \mathfrak{m})$, then $\lambda_m|_{U''_m} \equiv 0$ $(\mathrm{mod}\ \mathfrak{m})$.*

Proof. Let $\gamma_\bullet$ denote the differential of $C_\bullet$. We have $C_i = S_i \oplus T_i$,

$$\gamma_i = \begin{pmatrix} \varphi_i & 0 \\ \lambda_i & -\psi_i \end{pmatrix},$$

and $S_i \cong (S_i/U_i) \oplus U'_i \oplus U''_i \oplus U'''_i$ for each $i \in \mathbf{Z}$. Let (α_1, α_2) $(\alpha_1 \in S_m,\ \alpha_2 \in T_m)$ be an element of C_m such that $\gamma_m(\alpha_1, \alpha_2) \in U''_{m-1}$. Since $\varphi_m(\alpha_1) \in U''_{m-1}$, it follows from the hypothesis $H_m(S_\bullet/(U''_\bullet \oplus U'''_\bullet)) = 0$ that there is a $\beta \in S_{m+1}$ satisfying $\alpha_1 - \varphi_{m+1}(\beta) \in U''_m \oplus U'''_m$. Let $b_3 \in U''_m$ and $b_4 \in U'''_m$ be elements such that $\alpha_1 - \varphi_{m+1}(\beta) = (0, 0, b_3, b_4) \in S_m$. Then, $(\alpha_1, \alpha_2) - \gamma_{m+1}(\beta, 0) = ((0, 0, b_3, b_4), b_5)$ with some $b_5 \in T_m$, so that $(\alpha_1, \alpha_2) = \gamma_{m+1}(\beta, 0) + ((0, 0, b_3, b_4), b_5)$. This means that

$$\operatorname{Im}(\gamma_m) \cap U''_{m-1} \subset \operatorname{Im}(\gamma_m|_{U''_m \oplus U'''_m \oplus T_m}) \cap U''_{m-1}.$$

Now, since $U''_\bullet$ is a subcomplex of $S_\bullet$ and since $U''_{m-2} = 0$, $T_{m-2} = 0$ by hypotheses, the free submodule U''_{m-1} of C_{m-1} is contained in $\operatorname{Ker}(\gamma_{m-1})$. Suppose that $\mathfrak{m}H_{m-1}(C_\bullet) = 0$ and that $\lambda_m|_{U'''_m} \equiv 0$ $(\mathrm{mod}\ \mathfrak{m})$. Then, $\operatorname{Im}(\gamma_m)$ must contain $\mathfrak{m}U''_{m-1}$, namely, $\mathfrak{m}U''_{m-1} = \operatorname{Im}(\gamma_m) \cap \mathfrak{m}U''_{m-1}$. By the above observation, therefore,

$$\mathfrak{m}U''_{m-1} \subset \operatorname{Im}(\gamma_m|_{U''_m \oplus U'''_m \oplus T_m}) \cap U''_{m-1}. \tag{1}$$

Since $U'''_{m-1} = 0$ by hypotheses, $\operatorname{Im}(\gamma_m|_{U''_m \oplus U'''_m \oplus T_m}) \subset U''_{m-1} \oplus T_{m-1}$, so that we may write

$$\gamma_m|_{U''_m \oplus U'''_m \oplus T_m} = \begin{pmatrix} \varphi''_m & 0 & 0 \\ \lambda_m|_{U''_m} & \lambda_m|_{U'''_m} & -\psi_m \end{pmatrix},$$

where $\varphi''_\bullet$ denote the differential of $U''_\bullet$. Moreover,

$$\operatorname{Im}(\gamma_m|_{U''_m\oplus U'''_m\oplus T_m}) \cap U''_{m-1} = \bigl(\varphi''_m, 0, 0\bigr)(V)$$
$$\text{with } V := \operatorname{Ker}\bigl((\lambda_m|_{U''_m}, \lambda_m|_{U'''_m}, -\psi_m)\bigr). \tag{2}$$

Let $Z := (\varphi''_m, 0, 0)(V)$ and $Z' := \mathrm{pr}(V)$ for simplicity, where pr denotes the natural projection to U''_m. Since $\varphi''_m(U''_m) = \mathfrak{m}U''_{m-1}$ by hypothesis, we find by (1) and (2) that $\varphi''_m(Z') = Z = \mathfrak{m}U''_{m-1}$. Moreover, since $\varphi''_m(U''_m) = \varphi''_m(Z') = \mathfrak{m}U''_{m-1}$ and the homomorphism $\bar{\varphi}''_m : U''_m \otimes k \to U''_{m-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ induced from φ''_m is injective by hypotheses, we have $Z' + \mathfrak{m}U''_m = U''_m$, i.e., $Z' = U''_m$ by Nakayama's lemma. On the other hand, since $\psi_m \equiv 0 \pmod{\mathfrak{m}}$ by the minimality of $T_\bullet$ and since $\lambda_m|_{U'''_m} \equiv 0 \pmod{\mathfrak{m}}$ by assumption, it follows from the definitions of V and Z' that $\lambda_m(U''_m) = \lambda_m(Z') \subset \operatorname{Im}((\lambda_m|_{U'''_m}, -\psi_m)) \subset \mathfrak{m}T_{m-1}$. Hence $\lambda_m|_{U''_m} \equiv 0 \pmod{\mathfrak{m}}$. □

Lemma 1.7. *Let $(S_\bullet, \varphi_\bullet)$ be a free complex, $U_\bullet$ its free subcomplex with $S_\bullet/U_\bullet$ free, $(T_\bullet, \psi_\bullet)$ a minimal free complex with $T_i = 0$ for $i < m-1$ satisfying $\mathfrak{m}H_{m-1}(T_\bullet) = 0$, $H_i(T_\bullet) = 0$ for $i > m-1$ and $\lambda_\bullet : S_\bullet \to T_{\bullet-1}$ a chain map. If $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$, then there is a chain map $\lambda'_\bullet : S_\bullet \to T_{\bullet-1}$ satisfying $\lambda'_\bullet|_{U_\bullet} = 0$ which is chain equivalent to $\lambda_\bullet$.*

Proof. Note first that $H_{m-1}(T_\bullet) \cong T_{m-1}/\mathfrak{m}T_{m-1}$ and

$$\cdots \to T_m \to T_{m-1} \to H_{m-1}(T_\bullet) \to 0$$

is a minimal free resolution of $H_{m-1}(T_\bullet)$. Let $e_1, \ldots, e_b$ be a free basis of S_m such that $U_m = Ae_1 + \cdots + Ae_a$ $(0 \leqslant a \leqslant b)$. We first define a homomorphism $\lambda'_m : S_m \to T_{m-1}$ over A setting $\lambda'_m(e_i) = 0$ for $0 \leqslant i \leqslant a$ and $\lambda'_m(e_i) = \lambda_m(e_i)$ for $a < i \leqslant b$. Since $\lambda_m \circ \varphi_{m+1}(S_{m+1}) = \psi_m \circ \lambda_{m+1}(S_{m+1}) \subset \mathfrak{m}T_{m-1}$, for any element $u = \sum_{i=1}^b f_i e_i \in \operatorname{Im}(\varphi_{m+1})$, we have $\lambda'_m(u) = \sum_{i=1}^b f_i \lambda'_m(e_i) = \lambda_m(u) - \sum_{i=1}^a f_i \lambda_m(e_i) \in \mathfrak{m}T_{m-1}$ by hypothesis. Besides, $\lambda'_m \circ \varphi_{m+1}(U_{m+1}) \subset \lambda'_m(U_m) = 0$. There is therefore a homomorphism $\lambda'_{m+1} : S_{m+1} \to T_m$ such that $\lambda'_m \circ \varphi_{m+1} = \psi_m \circ \lambda'_{m+1}$ and $\lambda'_{m+1}|_{U_{m+1}} = 0$. Since $H_i(T_\bullet) = 0$ $(i > m-1)$ and $U_\bullet$ is a subcomplex of $S_\bullet$, one can construct successively homomorphisms $\lambda'_i : S_i \to T_{i-1}$ satisfying $\lambda'_i|_{U_i} = 0$ $(i \geqslant m+2)$ so that $\lambda'_\bullet : S_\bullet \to T_{\bullet-1}$ becomes a chain map, where $\lambda'_i = 0$ for $i < m$. Since $(\lambda_m - \lambda'_m)(e_i) \in \mathfrak{m}T_{m-1}$ for all $0 \leqslant i \leqslant b$ and since $T_\bullet$ is exact with $\psi_m(T_m) = \mathfrak{m}T_{m-1}$, the chain map $\lambda_\bullet - \lambda'_\bullet$ is chain homotopic to zero. Thus $\lambda'_\bullet$ fulfills all our requirements. □

Let m be an integer with $1 \leqslant m \leqslant d$ and p_l $(0 \leqslant l \leqslant m-1)$ nonnegative integers. We define

$$\begin{cases} U_\bullet := \bigoplus_{l=0}^{m-2} (K_{\bullet-l})^{p_l}, \qquad W_\bullet := \bigoplus_{l=0}^{m-1} (K_{\bullet-l})^{p_l}, \\ U'^{i,n}_\bullet := \bigoplus_{l=0}^{m-2} \bigl({K'_{\bullet-l}}^{i-l,n}\bigr)^{p_l}, \qquad U''^{i,n}_\bullet := \bigoplus_{l=0}^{m-2} \bigl({K''_{\bullet-l}}^{i-l,n}\bigr)^{p_l}, \\ \qquad U'''^{i,n}_\bullet := \bigoplus_{l=0}^{m-2} \bigl({K'''_{\bullet-l}}^{i-l,n}\bigr)^{p_l}, \\ W'^{i,n}_\bullet := \bigoplus_{l=0}^{m-1} \bigl({K'_{\bullet-l}}^{i-l,n}\bigr)^{p_l}, \qquad W''^{i,n}_\bullet := \bigoplus_{l=0}^{m-1} \bigl({K''_{\bullet-l}}^{i-l,n}\bigr)^{p_l}, \\ \qquad W'''^{i,n}_\bullet := \bigoplus_{l=0}^{m-1} \bigl({K'''_{\bullet-l}}^{i-l,n}\bigr)^{p_l} \end{cases} \tag{$*$}$$

for $0 \leqslant n \leqslant d$, $(x_1, \dots, x_r) \in \mathfrak{B}$ and $i \in \mathbf{Z}$, where $U_\bullet = U''^{i,n}_\bullet = U'''^{i,n}_\bullet = 0$ in case $m = 1$. Notice that

$$_{(n)}U_\bullet = U'^{i,n}_\bullet \oplus U''^{i,n}_\bullet \oplus U'''^{i,n}_\bullet, \qquad _{(n)}W_\bullet = W'^{i,n}_\bullet \oplus W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet.$$

Lemma 1.8. *Let $(S_\bullet, \varphi_\bullet)$ be a free complex which contains $U_\bullet$ as a free subcomplex such that $S_\bullet/U_\bullet$ is free. Let further $\lambda_\bullet : S_\bullet \to (L_{\bullet-m})^{p_{m-1}}$ be a chain map and $C_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$ its mapping cone. If $H_m({}_{(n)}S_\bullet/(U''^{m,n}_\bullet \oplus U'''^{m,n}_\bullet)) = 0$ and $\mathfrak{m}H_{m-1}({}_{(n)}C_\bullet) = 0$ for all $0 \leqslant n < d$ and $(x_1, \dots, x_r) \in \mathfrak{B}$, then $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$.*

Proof. Since $U_m = \bigoplus_{l=0}^{m-2}(K_{m-l})^{p_l}$, our assertion is equivalent to that $\lambda_m|_{(K_{m-l})^{p_l}} \equiv 0 \pmod{\mathfrak{m}}$ for all $0 \leqslant l \leqslant m-2$. If $\lambda_m|_{U_m} \not\equiv 0 \pmod{\mathfrak{m}}$, then there exists an l_0 $(0 \leqslant l_0 \leqslant m-2)$ such that $\lambda_m|_{(K_{m-l})^{p_l}} \equiv 0 \pmod{\mathfrak{m}}$ for all $l > l_0$ but $\lambda_m|_{(K_{m-l_0})^{p_{l_0}}} \not\equiv 0 \pmod{\mathfrak{m}}$. Note that $1 \leqslant m - l_0 - 1 < d$. Since $m - l > m - l_0 - 1$ for $l \leqslant l_0$, we see

$$U'''^{m,m-l_0-1}_\bullet = \bigoplus_{l=l_0+1}^{m-2} \bigl(K'''^{\,m-l,m-l_0-1}_{\bullet-l}\bigr)^{p_l} \subset \bigoplus_{l=l_0+1}^{m-2} ({}_{(m-l_0-1)}K_{\bullet-l})^{p_l},$$

so that ${}_{(m-l_0-1)}\lambda_m|_{U'''^{m,m-l_0-1}_m} \equiv 0 \pmod{\mathfrak{m}}$ for all $(x_1, \dots, x_r) \in \mathfrak{B}$ by the relation $\lambda_m|_{(K_{m-l})^{p_l}} \equiv 0 \pmod{\mathfrak{m}}$ $(l > l_0)$. On the other hand, since

$$K_{m-l_0} = \bigoplus_{1 \leqslant s_1 < \cdots < s_{m-l_0} \leqslant r} A \cdot \chi_{s_1} \wedge \cdots \wedge \chi_{s_{m-l_0}},$$

it follows from the condition $\lambda_m|_{(K_{m-l_0})^{p_{l_0}}} \not\equiv 0 \pmod{\mathfrak{m}}$ that $\lambda_m|_{(K''_{m-l_0})^{p_{l_0}}} \not\equiv 0 \pmod{\mathfrak{m}}$ after a permutation of $x_1, \dots, x_r$ if necessary, where

$$K''_{m-l_0} := K(x_1, \dots, x_{r-m+l_0+1})_1 \wedge \chi_{r-m+l_0+2} \wedge \cdots \wedge \chi_r.$$

In other words, there is a $(x_1, \dots, x_r) \in \mathfrak{B}$ such that $\lambda_m|_{(K''_{m-l_0})^{p_{l_0}}} \not\equiv 0 \pmod{\mathfrak{m}}$. Choose and fix such a $(x_1, \dots, x_r)$. The module $({}_{(m-l_0-1)}K''_{m-l_0})^{p_{l_0}} = (K''^{\,m-l_0,m-l_0-1}_{m-l_0})^{p_{l_0}}$ is a submodule of $U''^{m,m-l_0-1}_m$, therefore ${}_{(m-l_0-1)}\lambda_m|_{U''^{m,m-l_0-1}_m} \not\equiv 0 \pmod{\mathfrak{m}}$. We want to apply Lemma 1.6 to the complexes $({}_{(m-l_0-1)}S_\bullet, {}_{(m-l_0-1)}\varphi_\bullet)$, $({}_{(m-l_0-1)}L_{\bullet-(m-1)})^{p_{m-1}}$, ${}_{(m-l_0-1)}U_\bullet$, $U'^{m,m-l_0-1}_\bullet$, $U''^{m,m-l_0-1}_\bullet$, $U'''^{m,m-l_0-1}_\bullet$, and to the mapping cone ${}_{(m-l_0-1)}C_\bullet$ of the chain map ${}_{(m-l_0-1)}\lambda_\bullet$ over the local ring ${}_{(m-l_0-1)}A$ with maximal ideal $\mathfrak{m}/(x_{r-m+l_0+2}, \dots, x_r)$. Let us verify that they satisfy the hypotheses of that lemma. In fact, we see

$$({}_{(m-l_0-1)}L_{(i-2)-(m-1)})^{p_{m-1}} = 0,$$

$$U''^{m,m-l_0-1}_{i-2} = \bigoplus_{l=0}^{m-2} \bigl(K''^{\,m-l,m-l_0-1}_{i-2-l}\bigr)^{p_l} = 0$$

and

$$U'''_{i-1}{}^{m,m-l_0-1} = \bigoplus_{l=0}^{m-2} \left(K'''_{i-1-l}{}^{m-l,m-l_0-1}\right)^{p_l} = 0$$

for $i \leqslant m$, since $(i-2)-(m-1)<0$, $i-2-l<(m-l)-1$ and $i-1-l<m-l$, respectively. On the other hand, $U''_m{}^{m,m-l_0-1}$ and $U''_{m-1}{}^{m,m-l_0-1}$ are the direct sums of p_l $(l_0 \leqslant l \leqslant m-2)$ copies of

$$\bigoplus_{r-m+l_0+2 \leqslant s_1 < \cdots < s_{m-l-1} \leqslant r} {}_{(m-l_0-1)}K(x_1,\ldots,x_{r-m+l_0+1})_1 \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_{m-l-1}}$$

and

$$\bigoplus_{r-m+l_0+2 \leqslant s_1 < \cdots < s_{m-l-1} \leqslant r} {}_{(m-l_0-1)}K(x_1,\ldots,x_{r-m+l_0+1})_0 \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_{m-l-1}},$$

respectively, with the differential ${}_{(m-l_0-1)}\varphi_m|_{U''_m{}^{m,m-l_0-1}}$ being the direct sum of some copies of the usual first differential of ${}_{(m-l_0-1)}K(x_1,\ldots,x_{r-m+l_0+1})_\bullet$. The conditions on ${}_{(m-l_0-1)}\varphi_m$ are therefore valid. As we have already seen, ${}_{(m-l_0-1)}\lambda_m|_{U'''_m{}^{m,m-l_0-1}} \equiv 0 \pmod{\mathfrak{m}}$. Moreover, by our hypotheses of the present lemma, $\mathfrak{m}H_{m-1}({}_{(m-l_0-1)}C_\bullet)=0$ and $H_m({}_{(m-l_0-1)}S_\bullet/(U''_\bullet{}^{m,m-l_0-1} \oplus U'''_\bullet{}^{m,m-l_0-1}))=0$. Thus ${}_{(m-l_0-1)}\lambda_m|_{U''_m{}^{m,m-l_0-1}} \equiv 0 \pmod{\mathfrak{m}}$ by Lemma 1.6, which is a contradiction. Hence $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$. $\square$

Lemma 1.9. *Let $S_\bullet$ be a free complex which contains $U_\bullet$ as a subcomplex such that $S_\bullet/U_\bullet$ is free. Let further $\lambda_\bullet : S_\bullet \to (L_{\bullet-m})^{p_{m-1}}$ be a chain map satisfying $\lambda_\bullet|_{U_\bullet}=0$ and $C_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$ its mapping cone. Suppose that, for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, $(x_1,\ldots,x_r) \in \mathfrak{B}$, the homomorphism $H_i(U'''_\bullet{}^{i,n}) \to H_i({}_{(n)}S_\bullet)$ induced from the composition of inclusions $U'''_\bullet{}^{i,n} \hookrightarrow {}_{(n)}U_\bullet \hookrightarrow {}_{(n)}S_\bullet$ is surjective and $H_i({}_{(n)}S_\bullet/(U''_\bullet{}^{i,n} \oplus U'''_\bullet{}^{i,n}))=0$. Then $C_\bullet$ contains $W_\bullet$ as a subcomplex with $C_\bullet/W_\bullet$ free, and moreover the homomorphism $H_i(W'''_\bullet{}^{i,n}) \to H_i({}_{(n)}C_\bullet)$ induced from the composition of inclusions $W'''_\bullet{}^{i,n} \hookrightarrow {}_{(n)}W_\bullet \hookrightarrow {}_{(n)}C_\bullet$ is surjective and $H_i({}_{(n)}C_\bullet/(W''_\bullet{}^{i,n} \oplus W'''_\bullet{}^{i,n}))=0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, $(x_1,\ldots,x_r) \in \mathfrak{B}$.*

Proof. Put

$$T_\bullet := (L_{\bullet-(m-1)})^{p_{m-1}}, \qquad V_\bullet := (K_{\bullet-(m-1)})^{p_{m-1}},$$
$$V'_\bullet{}^{i,n} := \left(K'_{\bullet-(m-1)}{}^{i-(m-1),n}\right)^{p_{m-1}}, \qquad V''_\bullet{}^{i,n} := \left(K''_{\bullet-(m-1)}{}^{i-(m-1),n}\right)^{p_{m-1}},$$
$$V'''_\bullet{}^{i,n} := \left(K'''_{\bullet-(m-1)}{}^{i-(m-1),n}\right)^{p_{m-1}}.$$

Then $V_\bullet$ is a subcomplex of $T_\bullet$ such that $T_\bullet/V_\bullet$ is free. Since $C_j = S_j \oplus T_j$ and $\lambda_j|_{U_j}=0$ for all $j \in \mathbf{Z}$, $W_\bullet = U_\bullet \oplus V_\bullet$ is a subcomplex of $C_\bullet$ such that $C_\bullet/W_\bullet$ is free. By this and our hypotheses, ${}_{(n)}U_\bullet$ and ${}_{(n)}V_\bullet$ are subcomplexes of ${}_{(n)}S_\bullet$ and ${}_{(n)}T_\bullet$ respectively with

${}_{(n)}S_\bullet/{}_{(n)}U_\bullet$ and ${}_{(n)}T_\bullet/{}_{(n)}V_\bullet$ being free over ${}_{(n)}A$, ${}_{(n)}C_\bullet$ is the mapping cone of the chain map ${}_{(n)}\lambda_\bullet : {}_{(n)}S_\bullet \to {}_{(n)}T_{\bullet-1}$, ${}_{(n)}\lambda_\bullet|_{{}_{(n)}U_\bullet} = 0$, and ${}_{(n)}W_\bullet = {}_{(n)}U_\bullet \oplus {}_{(n)}V_\bullet$ is a subcomplex of ${}_{(n)}C_\bullet$ such that ${}_{(n)}C_\bullet/{}_{(n)}W_\bullet$ is free over ${}_{(n)}A$. Moreover ${}_{(n)}V_\bullet = V'^{i,n}_\bullet \oplus V''^{i,n}_\bullet \oplus V'''^{i,n}_\bullet$. The exact sequence

$$0 \to {}_{(n)}T_\bullet \to {}_{(n)}C_\bullet \to {}_{(n)}S_\bullet \to 0$$

yields an exact sequence

$$\begin{aligned} 0 \to {}_{(n)}T_\bullet/\bigl(V''^{i,n}_\bullet \oplus V'''^{i,n}_\bullet\bigr) &\to {}_{(n)}C_\bullet/\bigl(W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet\bigr) \\ &\to {}_{(n)}S_\bullet/\bigl(U''^{i,n}_\bullet \oplus U'''^{i,n}_\bullet\bigr) \to 0, \end{aligned}$$

since $W''^{i,n}_\bullet = U''^{i,n}_\bullet \oplus V''^{i,n}_\bullet$ and $W'''^{i,n}_\bullet = U'''^{i,n}_\bullet \oplus V'''^{i,n}_\bullet$. Taking the long exact sequence, we find that $H_i({}_{(n)}C_\bullet/(W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet)) = 0$, since $H_i({}_{(n)}T_\bullet/(V''^{i,n}_\bullet \oplus V'''^{i,n}_\bullet)) = 0$ by Lemma 1.5 and $H_i({}_{(n)}S_\bullet/(U''^{i,n}_\bullet \oplus U'''^{i,n}_\bullet)) = 0$ by hypotheses. On the other hand, since the direct sum $W'''^{i,n}_\bullet = U'''^{i,n}_\bullet \oplus V'''^{i,n}_\bullet$ is also a subcomplex of ${}_{(n)}C_\bullet$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & {}_{(n)}T_\bullet \arrow[r] & {}_{(n)}C_\bullet \arrow[r] & {}_{(n)}S_\bullet \arrow[r] & 0 \\ 0 \arrow[r] & V'''^{i,n}_\bullet \arrow[r] \arrow[u] & W'''^{i,n}_\bullet \arrow[r] \arrow[u] & U'''^{i,n}_\bullet \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

of short exact sequences, where the bottom one splits. The homomorphism $H_i(V'''^{i,n}_\bullet) \to H_i({}_{(n)}T_\bullet)$ is bijective by Lemma 1.4 and the homomorphism $H_i(U'''^{i,n}_\bullet) \to H_i({}_{(n)}S_\bullet)$ is surjective by hypotheses. We can therefore verify the surjectivity of the homomorphism $H_i(W'''^{i,n}_\bullet) \to H_i({}_{(n)}C_\bullet)$ again by taking the long exact sequence. □

Lemma 1.10. *Let $F_\bullet$ be a free complex satisfying $H_i(F_\bullet) = 0$ for all $i \in \mathbf{Z}$. Then $H_i({}_{(n)}F_\bullet) = 0$, $H_i({}_{(n)}F^\vee_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$.*

Proof. Let $\partial_\bullet$ denote the differential of $F_\bullet$. For each l, the sequence

$$\cdots \to F_{l+d} \to \cdots \to F_{l+1} \to F_l \to \mathrm{Im}(\partial_l) \to 0$$

is a free resolution of $\mathrm{Im}(\partial_l)$. Since A is Gorenstein of dimension d, we have $\mathrm{Ext}^j_A(\mathrm{Im}(\partial_l), A) = 0$ for all $j > d$. This being true for all $l \in \mathbf{Z}$, the dual $F^\vee_\bullet$ is also an exact complex. Hence $\mathrm{Ext}^j_A(\mathrm{Im}(\partial_l), A) = 0$ for all $j > 0$ and $\mathrm{Im}(\partial_l)$ is maximal Cohen–Macaulay. Let $(x_1, \ldots, x_r) \in \mathfrak{B}$. Since the sequence $x_{r-d+1}, \ldots, x_r$ forms a system of parameters of A, it is also a system of parameters of $\mathrm{Im}(\partial_l)$ for all $l \in \mathbf{Z}$. Hence $\mathrm{Tor}^A_j(\mathrm{Im}(\partial_l), {}_{(n)}A) = 0$ for all $j > 0$, $0 \leqslant n \leqslant d$, $l \in \mathbf{Z}$. Thus $H_i({}_{(n)}F_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$. Since $F^\vee_\bullet$ is exact as we have already seen, it follows in the same way that $H_i({}_{(n)}F^\vee_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$. □

Now we can proceed to our main results of this section.

Proposition 1.11. *Let $E_\bullet$ be a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. Let further $W_\bullet$, $W''^{i,n}_\bullet$ and $W'''^{i,n}_\bullet$ be subcomplexes of $E_\bullet$ as in $(*)$. Then the natural homomorphism $H_i(W'''^{i,n}_\bullet) \to H_i({}_{(n)}E_\bullet)$ is surjective and $H_i({}_{(n)}E_\bullet/(W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet)) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$.*

Proof. Let $S_\bullet$ denote $\operatorname{con}(\lambda_{m-2,\bullet})_\bullet$ for $m \geqslant 2$ and $F_\bullet$ for $m = 1$. Let further $U_\bullet$, $U''^{i,n}_\bullet$, $U'''^{i,n}_\bullet$ be as in $(*)$, where $U_\bullet = U''^{i,n}_\bullet = U'''^{i,n}_\bullet = 0$ in case $m = 1$. Suppose that $H_i({}_{(n)}S_\bullet/(U''^{i,n}_\bullet \oplus U'''^{i,n}_\bullet)) = 0$ and that the natural homomorphism $H_i(U'''^{i,n}_\bullet) \to H_i({}_{(n)}S_\bullet)$ is surjective for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$. Then, since $E_\bullet$ is the mapping cone of $\lambda_{m-1,\bullet} : S_\bullet \to (L_{\bullet-m})^{p_{m-1}}$ with $\lambda_{m-1,\bullet}|_{U_\bullet} = 0$, we see by Lemma 1.9 that $H_i({}_{(n)}E_\bullet/(W''^{i,n}_\bullet \oplus W'''^{i,n}_\bullet)) = 0$ and that the natural homomorphism $H_i(W'''^{i,n}_\bullet) \to H_i({}_{(n)}E_\bullet)$ is surjective for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$. Our assertion therefore follows by induction on m, since $U''^{i,n}_\bullet = U'''^{i,n}_\bullet = 0$ and $H_i({}_{(n)}F_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$ by Lemma 1.10 in the initial case $m = 1$. $\square$

Corollary 1.12. *Let $E_\bullet$ be a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. Then $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$.*

Proof. Since $W'''^{i,n}_j = 0$ for $j < i$ and

$$W'''^{i,n}_\bullet = \bigoplus_{l=0}^{m-1} \left(\bigoplus_{r-n+1 \leqslant s_1 < \cdots < s_{i-l} \leqslant r} {}_{(n)}K(x_1, \ldots, x_{r-n})_{\bullet-i} \wedge \chi_{s_1} \wedge \cdots \wedge \chi_{s_{i-l}} \right)^{p_l},$$

we see $\mathfrak{m}H_i(W'''^{i,n}_\bullet) = 0$. Hence $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ by the preceding proposition. $\square$

Proposition 1.13. *Let $E_\bullet$ be a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-1})$. If $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ for all $0 \leqslant i < m$, $0 \leqslant n < d$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$, then $E_\bullet$ is chain isomorphic to a Buchsbaum cone with the same base.*

Proof. If $m = 1$, there is nothing to prove. Suppose that $m \geqslant 2$ and that our assertion is true for $m - 1$. Denote $\operatorname{con}(\lambda_{m-2,\bullet})_\bullet$ by $D_\bullet$. Since ${}_{(n)}E_\bullet$ is the mapping cone of ${}_{(n)}\lambda_{m-1,\bullet}$ for $0 \leqslant n < d$, there is an exact sequence

$$0 \to {}_{(n)}L_{\bullet-(m-1)} \to {}_{(n)}E_\bullet \to {}_{(n)}D_\bullet \to 0$$

that yields a long exact sequence

$$\cdots \to H_i({}_{(n)}E_\bullet) \to H_i({}_{(n)}D_\bullet) \to H_{i-1}({}_{(n)}L_{\bullet-(m-1)}) \to \cdots.$$

Here $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ and ${}_{(n)}L_{(i-1)-(m-1)} = 0$ for all $0 \leqslant i < m$, so that $\mathfrak{m}H_i({}_{(n)}D_\bullet) = 0$ for all $0 \leqslant i < m - 1$ and $(x_1, \ldots, x_r) \in \mathfrak{B}$. Since $D_\bullet$ is a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{m-2})$, it is chain isomorphic to a Buchsbaum cone with base

$(F_\bullet, p_0, \dots, p_{m-2})$ by the induction hypothesis, say $S_\bullet$. By Proposition 1.11, this $S_\bullet$ satisfies $H_m({}_{(n)}S_\bullet/(U''^{m,n}_\bullet \oplus U'''^{m,n}_\bullet)) = 0$ with the notation of $(*)$. Let $\nu_\bullet : S_\bullet \to D_\bullet$ be the isomorphism mentioned above, $\lambda_\bullet := \lambda_{m-1,\bullet} \circ \nu_\bullet$, and $C_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$. Since $C_\bullet = \mathrm{con}(\lambda_\bullet)_\bullet \cong \mathrm{con}(\lambda_{m-1,\bullet})_\bullet = E_\bullet$, we have $\mathfrak{m}H_{m-1}({}_{(n)}C_\bullet) = 0$ for all $0 \leqslant n < d$ and $(x_1, \dots, x_r) \in \mathfrak{B}$ by hypothesis. Hence $\lambda_m|_{U_m} \equiv 0 \pmod{\mathfrak{m}}$ by Lemma 1.8. Finally, let $\lambda'_\bullet : S_\bullet \to (L_{\bullet-m})^{p_{m-1}}$ be the chain map satisfying $\lambda'_\bullet \simeq \lambda_\bullet$ and $\lambda'_\bullet|_{U_\bullet} = 0$ as in Lemma 1.7. Then the mapping cone $\mathrm{con}(\lambda'_\bullet)_\bullet$ is a Buchsbaum cone with base $(F_\bullet, p_0, \dots, p_{m-1})$ which is isomorphic to $C_\bullet \cong E_\bullet$. □

Finally, we prove a theorem that plays a crucial role in the proof our main result of this paper presented in the next section. Recall that, given a free complex $(S_\bullet, \varphi_\bullet)$ satisfying $\mathrm{Im}(\varphi_i) \subset \mathfrak{m}S_{i-1}$ for all $i \leqslant 0$, there are a minimal free complex $\min(S_\bullet)_\bullet$ and a split exact free complex $\mathrm{se}(S_\bullet)_\bullet$ such that $S_\bullet \cong \min(S_\bullet)_\bullet \oplus \mathrm{se}(S_\bullet)_\bullet$. Moreover $\min(S_\bullet)_\bullet$ and $\mathrm{se}(S_\bullet)_\bullet$ are determined uniquely by this condition up to isomorphism (see [2, (1.1) and (1.2)]).

Theorem 1.14. *Let m be an integer with $1 \leqslant m \leqslant d$ and let $G_\bullet$ be a minimal free complex satisfying $H^i(G_\bullet^\vee) = 0$ for $i < d$, $H_i(G_\bullet) = 0$ for $i < 0$, $i \geqslant m$, and $\mathfrak{m}H_i({}_{(n)}G_\bullet) = 0$ for all $0 \leqslant i < m$, $0 \leqslant n < d$, $(x_1, \dots, x_r) \in \mathfrak{B}$. Then there is a Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ such that $G_\bullet = \min(E_\bullet)_\bullet$, where $p_i = l_R(H_i(G_\bullet))$ $(0 \leqslant i \leqslant m-1)$.*

Proof. If $p_i = 0$ for all $0 \leqslant i \leqslant m-1$, then $G_\bullet$ is a Buchsbaum cone with base $(G_\bullet, 0, \dots, 0)$. Suppose that $p_i \neq 0$ for some $0 \leqslant i \leqslant m-1$. With the notation of [2, Section 1], let $F_\bullet := \sigma_0(G_\bullet)$. Then by [2, (1.6)] there is a quasi-Buchsbaum cone $E'_\bullet$ with base $(F_\bullet, p_0, \dots, p_{m-1})$ such that $G_\bullet = \min(E'_\bullet)_\bullet$. Since $H_i({}_{(n)}G_\bullet) \cong H_i({}_{(n)}E'_\bullet)$ for all $0 \leqslant n < d$, $i \in \mathbf{Z}$, and $(x_1, \dots, x_r) \in \mathfrak{B}$, we find by Proposition 1.13, that $E'_\bullet$ is isomorphic to a Buchsbaum cone $E_\bullet$ with the same base. Our assertion follows from the uniqueness of the minimal part of a free complex (see [2, (1.1)]). □

2. Structure theorem

In this section, applying the results obtained so far, we prove our main theorem which generalizes Goto's structure theorem for maximal Buchsbaum modules over regular local rings (see [5]). We continue to assume that $(A, \mathfrak{m})$ is a local Gorenstein ring of dimension $d > 0$ with $k := A/\mathfrak{m}$ and $r := \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. As in the preceding section, we fix an A-basis $z_1, \dots, z_r$ of $\mathfrak{m}$ and denote by $\mathfrak{B}$ the set of A-bases of $\mathfrak{m}$ obtained by permuting $z_1, \dots, z_r$.

Lemma 2.1. *Let $y_1, \dots, y_d$ be a system of parameters of A, $\bar{A} := A/(y_1)$, $\tilde{A} := \bar{A}/(y_2, \dots, y_n) = A/(y_1, \dots, y_n)$ for $1 \leqslant n < d$, and M a finitely generated module over A. If y_1 is not contained in any associated prime of M different from $\mathfrak{m}$, then $\mathrm{Ext}^1_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) \cong \mathrm{Ext}^1_A(M, \tilde{A})$.*

Proof. Let

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a minimal free resolution of M over A and let $M_i = \mathrm{Im}(\delta_i)$ $(i \geqslant 1)$. Tensoring this resolution with $\bar{A}$ over A, we obtain short exact sequences

$$0 \to N \to \bar{A} \otimes_A M_1 \to K \to 0, \tag{3}$$

$$0 \to K \to \bar{A} \otimes_A P_0 \to \bar{A} \otimes M \to 0, \tag{4}$$

where $K := \mathrm{Ker}(\bar{A} \otimes \varepsilon)$, $N := \mathrm{Tor}_1^A(\bar{A}, M)$. Since y_1 is not contained in any associated prime of M different from $\mathfrak{m}$, we have $N \subset H^0_{\mathfrak{m}}(M)$ and $\mathfrak{m}^l N = 0$ for $l \gg 0$. This, together with the fact $\mathrm{depth}_{\mathfrak{m}}(\tilde{A}) > 0$, shows that $\mathrm{Hom}_{\bar{A}}(N, \tilde{A}) = 0$. Hence

$$\mathrm{Hom}_{\bar{A}}(K, \tilde{A}) \cong \mathrm{Hom}_{\bar{A}}(\bar{A} \otimes_A M_1, \tilde{A}) = \mathrm{Hom}_A(M_1, \tilde{A})$$

by (3). It follows therefore from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\bar{A}}(\bar{A} \otimes P_0, \tilde{A}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{\bar{A}}(K, \tilde{A}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^1_{\bar{A}}(\bar{A} \otimes M, \tilde{A}) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_A(P_0, \tilde{A}) \arrow[r] & \mathrm{Hom}_A(M_1, \tilde{A}) \arrow[r] & \mathrm{Ext}^1_A(M, \tilde{A}) \arrow[r] & 0
\end{tikzcd}$$

that $\mathrm{Ext}^1_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) \cong \mathrm{Ext}^1_A(M, \tilde{A})$. $\square$

Lemma 2.2. *Let M be a maximal quasi-Buchsbaum module over A such that M/x_rM is Buchsbaum for all $(x_1, \ldots, x_r) \in \mathfrak{B}$. Then M is Buchsbaum.*

Proof. Let $(x_1, \ldots, x_r) \in \mathfrak{B}$, $\bar{A} := A/(x_r)$ and $\overline{M} := M/x_rM$. Observe that $z_1, \ldots, z_r$ form an M-basis of $\mathfrak{m}$ and that $x_{r-d+1}, \ldots, x_r$ (respectively $x_{r-d+1}, \ldots, x_{r-1}$) form a system of parameters of M (respectively $\overline{M}$). Moreover, $0 :_M x_r = H^0_{\mathfrak{m}}(M)$ (see, e.g., [3, Lemma (1.2)]). Since $\overline{M}$ is Buchsbaum over $\bar{A}$ with $\dim(\overline{M}) = d - 1$ by hypothesis, we have

$$l_A\bigl(\overline{M}/(x_{r-d+1}, \ldots, x_{r-1})\overline{M}\bigr) - \mathrm{e}_{(x_{r-d+1},\ldots,x_{r-1})\bar{A}}(\overline{M}) = \sum_{j=0}^{d-2} \binom{d-2}{j} l_A\bigl(H^j_{\mathfrak{m}}(\overline{M})\bigr). \tag{5}$$

Since $l_A(H^j_{\mathfrak{m}}(\overline{M})) = l_A(H^j_{\mathfrak{m}}(M)) + l_A(H^{j+1}_{\mathfrak{m}}(M))$ for $j \geqslant 0$ by the long exact sequences arising from

$$0 \to H^0_{\mathfrak{m}}(M) = 0 :_M x_r \to M \xrightarrow{x_r} M \to \overline{M} \to 0,$$

the equality (5) yields

$$l_A\bigl(M/(x_{r-d+1}, \ldots, x_r)M\bigr) - \mathrm{e}_{(x_{r-d+1},\ldots,x_r)A}(M) = \sum_{j=0}^{d-1} \binom{d-1}{j} l_A\bigl(H^j_{\mathfrak{m}}(M)\bigr).$$

This being true for all $(x_1, \dots, x_r) \in \mathfrak{B}$, it follows from [9, Chapter I, Theorem 2.15] and [9, Appendix, Theorem 20] that M is Buchsbaum. See [10, Corollary 2.4 and Proposition 3.2] also. $\square$

Theorem 2.3. *Let M be a maximal quasi-Buchsbaum module over A. Then M is Buchsbaum if and only if $\mathfrak{m}\,\mathrm{Ext}^i_A(M, A/(x_{r-n+1}, \dots, x_r)) = 0$ for all $i > 0$, $0 \leqslant n < d$, and $(x_1, \dots, x_r) \in \mathfrak{B}$.*

Proof. If $d = 1$, then M is Buchsbaum and there is nothing to prove. Assume that $d > 1$ and that our assertion is true for $d - 1$. Let

$$\cdots \xrightarrow{\delta_4} P_3 \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a minimal free resolution of M over A, $(x_1, \dots, x_r) \in \mathfrak{B}$, $\bar{A} := A/(x_r)$, $\tilde{A} := A/(x_{r-n+1}, \dots, x_r)$, $M_i = \mathrm{Im}(\delta_i)$ $(i \geqslant 1)$, $K := \mathrm{Ker}(\bar{A} \otimes \varepsilon)$, and $N := \mathrm{Tor}^A_1(\bar{A}, M)$. Then we have

$$\mathrm{Ext}^i_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) \cong \mathrm{Ext}^{i-1}_{\bar{A}}(K, \tilde{A}) \quad (i \geqslant 2), \tag{6}$$

$$\mathrm{Ext}^1_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) \cong \mathrm{Ext}^1_A(M, \tilde{A}), \tag{7}$$

$$\mathrm{Ext}^i_{\bar{A}}(\bar{A} \otimes_A M_1, \tilde{A}) \cong \mathrm{Ext}^i_A(M_1, \tilde{A}) \cong \mathrm{Ext}^{i+1}_A(M, \tilde{A}) \quad (i > 0). \tag{8}$$

The first relation is a consequence of the long exact sequence arising from (4), the second holds by Lemma 2.1, and the third follows from the fact that the complex

$$\cdots \xrightarrow{\delta_4} P_3 \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} M_1 \to 0$$

tensored with $\bar{A}$ is exact. Now if M is Buchsbaum, so are $\bar{A} \otimes_A M$, M_1 and $\bar{A} \otimes_A M_1$ (see, e.g., [9, Chapter I, Theorem 2.15]). By the induction hypothesis, we have $\mathfrak{m}\,\mathrm{Ext}^i_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) = 0$ for all $i > 0$, $1 \leqslant n < d$. It follows therefore from (7) and (8) that $\mathfrak{m}\,\mathrm{Ext}^i_A(M, \tilde{A}) = 0$ for all $i > 0$, $1 \leqslant n < d$. Since $\mathfrak{m}\,\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$ by the hypothesis that M is quasi-Buchsbaum, this proves the "only if" part. Conversely, suppose that $\mathfrak{m}\,\mathrm{Ext}^i_A(M, \tilde{A}) = 0$ for all $i > 0$, $0 \leqslant n < d$, and $(x_1, \dots, x_r) \in \mathfrak{B}$. The relation (8) shows that $\mathfrak{m}\,\mathrm{Ext}^i_{\bar{A}}(\bar{A} \otimes_A M_1, \tilde{A}) = 0$ for all $i > 0$, $1 \leqslant n < d$, and $(x_1, \dots, x_r) \in \mathfrak{B}$. Since $\dim(\bar{A} \otimes_A M_1) = d - 1$ and $x_1, \dots, x_{r-1}$ form an $\bar{A}$-basis, we find that $\bar{A} \otimes_A M_1$ is Buchsbaum by the induction hypothesis. Moreover, since $\mathrm{depth}_{\mathfrak{m}}(K) > 0$, it follows from the exact sequence (3) that $H^0_{\mathfrak{m}}(\bar{A} \otimes M_1) = H^0_{\mathfrak{m}}(N) = N \subset H^0_{\mathfrak{m}}(M)$. Consequently, $\mathfrak{m}H^0_{\mathfrak{m}}(\bar{A} \otimes_A M_1) = 0$ and K is Buchsbaum (see, e.g., [9, Chapter I, Proposition 2.22]). Thus $\mathfrak{m}\,\mathrm{Ext}^i_{\bar{A}}(\bar{A} \otimes_A M, \tilde{A}) = 0$ for all $i > 0$, $1 \leqslant n < d$, and $(x_1, \dots, x_r) \in \mathfrak{B}$ by (6) and (7). Again by the induction hypothesis, we see that $\bar{A} \otimes_A M$ is Buchsbaum. Since M is quasi-Buchsbaum by hypothesis and $\bar{A} \otimes_A M = M/x_r M$ is Buchsbaum as already seen, with x_r not contained in any associated prime of M different from $\mathfrak{m}$ for all $(x_1, \dots, x_r) \in \mathfrak{B}$, we conclude that M is Buchsbaum by Lemma 2.2. Thus the "if" part also holds. $\square$

Theorem 2.4. *Let M be a finitely generated maximal module over A having no free direct summand,*

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

a minimal free resolution of M over A and

$$\cdots \xrightarrow{\delta_{-2}^\vee} P_{-2}^\vee \xrightarrow{\delta_{-1}^\vee} P_{-1}^\vee \xrightarrow{\delta_0^\vee} P_0^\vee \xrightarrow{\delta_1^\vee} \mathrm{Im}(\delta_1^\vee) \to 0,$$

a minimal free resolution of $\mathrm{Im}(\delta_1^\vee)$ *over A. We put $G_i = P_{d-i}^\vee$, $\gamma_i = \delta_{d-i+1}^\vee$ for $i \in \mathbf{Z}$ and let $(G_\bullet, \gamma_\bullet)$ be the minimal complex thus obtained* (*cf.* [2, (4.2)]). *Then M is Buchsbaum if and only if $G_\bullet$ is the minimal part of a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{d-1})$ for some minimal free exact complex $F_\bullet$ and nonnegative integers $p_0, \ldots, p_{d-1}$. Moreover when this is the case, we have $p_i = l_A(H_{\mathfrak{m}}^i(M))$ for all $0 \leqslant i < d$.*

Proof. First of all, by definition, $H^i(G_\bullet^\vee) = 0$ for $i < d$, $H_i(G_\bullet) = \mathrm{Ext}_A^{d-i}(M, A) = 0$ for $i < 0$ and $H_i(G_\bullet) = H^{d-i}(P_\bullet^\vee) = 0$ for $i \geqslant d$. Let $(x_1, \ldots, x_r) \in \mathfrak{B}$. We have $\mathfrak{m}H_{\mathfrak{m}}^i(M) = 0$ if and only if $\mathfrak{m}\,\mathrm{Ext}_A^{d-i}(M, A) = \mathfrak{m}H_i(G_\bullet) = 0$ for $i < d$ by local duality and $\mathrm{Ext}_A^{d-i}(M, {}_{(n)}A) = H_i({}_{(n)}G_\bullet)$ for $0 \leqslant i < d$, $0 \leqslant n < d$. Suppose that M is quasi-Buchsbaum. Then, since $\mathfrak{m}H_i(G_\bullet) = \mathfrak{m}\,\mathrm{Ext}_A^{d-i}(M, A) = 0$ for $i < d$ and $H_i(G_\bullet) = 0$ for $i < 0$, $i \geqslant d$ as we have already mentioned above, the complex $G_\bullet$ is the minimal part of a quasi-Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{d-1})$ for some minimal free exact complex $F_\bullet$ and nonnegative integers $p_i := l_A(H_i(G_\bullet)) = l_A(H_{\mathfrak{m}}^i(M))$ $(0 \leqslant i < d)$ by [2, (1.6)]. If further M is Buchsbaum, then $\mathfrak{m}H_i({}_{(n)}G_\bullet) = \mathfrak{m}\,\mathrm{Ext}_A^{d-i}(M, {}_{(n)}A) = 0$ for all $0 \leqslant i < d$, $0 \leqslant n < d$ by Theorem 2.3. Hence $G_\bullet$ is the minimal part of a Buchsbaum cone with base $(F_\bullet, p_0, \ldots, p_{d-1})$ by Theorem 1.14 as desired. Since a Buchsbaum cone $E_\bullet$ with base $(F_\bullet, p_0, \ldots, p_{d-1})$ satisfies $\mathfrak{m}H_i({}_{(n)}E_\bullet) = 0$ for all $0 \leqslant n \leqslant d$, $i \in \mathbf{Z}$, and $(x_1, \ldots, x_r) \in \mathfrak{B}$ by Corollary 1.12, the converse also holds by Theorem 2.3. □

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